УДК 517.5

The boundedness and compactness of commutators singular integral operator on generalized Morrey spaces with variable exponent[1]

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We prove the boundedness and compactness of the commutators of the Calderón-Zygmund singular operators in variable exponent generalized Morrey spaces in case of unbounded sets $\Omega \subset \mathbb{R}^n$, where $b \in BMO(\Omega)$.

Keywords: Calderón-Zygmund singular operators, commutator, generalized Morrey space, BMO space.

Ограниченность и компактность коммутаторов сингулярного интегрального оператора на обобщенных пространствах Морри с переменным показателем степени[1]

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Доказывается ограниченность и компактность коммутаторов сингулярных операторов Кальдерона-Зигмунда в пространствах Морри с переменными показателями в случае неограниченных множеств $\Omega \subset \mathbb{R}^n$, где $b \in BMO(\Omega)$.

Ключевые слова: сингулярные операторы Кальдерона-Зигмунда, коммутатор, обобщенное пространство Морри, пространство BMO.

The classical Morrey spaces were originally introduced by Morrey in [1] to study the local behavior of solutions to second order elliptic partial differential equations. As it is known, last two decades there is an increasing interest to the study of variable exponent spaces and operators with variable parameters in such spaces, we refer for instance to the surveying papers [2,3], on the progress in this field, including topics of Harmonic Analysis and Operator Theory, see also references therein. For mapping properties of maximal functions and singular integrals on Lebesgue spaces with variable exponent we refer to [4–8].

Variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$, were introduced and studied in [9] in the Euclidean setting and in [10] in the setting of metric measure spaces, in case of bounded sets. The boundedness of the maximal operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the log-condition on $p(\cdot)$, $\lambda(\cdot)$ was proved in [9]. P. Hästö in [11] used his new "local-to-global"approach to extend the result of [9] on the maximal operator to the case of the whole space $\mathbb{R}^n$. The boundedness of the maximal operator and the singular integral operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$ in the general setting of metric measure spaces was proved in [10].

Generalized Morrey spaces of such a kind in the case of constant p were studied in [12,13]. In the case of bounded sets the boundedness of the maximal operator, singular integral operators and the potential operator in generalized variable exponent Morrey type spaces was proved in [14,15]. Also, in the case of bounded sets the boundedness of these operators in generalized variable exponent weighted Morrey spaces for the power weights was proved in [16–18].

We introduce the generalized variable exponent Morrey spaces $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$ over an open set $\Omega \subset e$. Within the frameworks of the spaces $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$, over unbounded sets $\Omega \subseteq \mathbb{R}^n$ we consider the Calderón-Zygmund type singular operator

$$Tf(x) = \int\limits_{\Omega} K(x,y)f(y)dy,$$

where $K(x,y)$ is a "standard singular kernel that is, a continuous function defined on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \leq C|x-y|^{-n} \text{ for all } x \neq y,$$

$$|K(x,y) - K(x,z)| \leq C\frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \text{ if } |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \leq C\frac{|x-\xi|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \text{ if } |x-y| > 2|x-\xi|.$$

Let

$$T^*f(x) = \sup_{\varepsilon>0} |T_{\varepsilon}f(x)|$$

be the maximal singular operator, where $T_{\varepsilon}f(x)$ is the usual truncation

$$T_{\varepsilon}f(x) = \int\limits_{\{y\in\Omega:|x-y|\geq\varepsilon\}} K(x,y)f(y)dy.$$

We find the condition on the Morrey function $\varphi(x,r)$ for the boundedness of the commutators of the commutators of the Riesz potential and Calderón-Zygmund singular operators in generalized Morrey space $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$ with variable $p(x)$ under the log-condition on $p(\cdot)$. Also we prove the compactness of the commutators of the Riesz potential and Calderón-Zygmund singular operators in variable exponent generalized Morrey spaces, where $b \in VMO(\Omega)$.

We use the following notation: $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\Omega \subset \mathbb{R}^n$ is an open set, $\chi_E(x)$ is the characteristic function of a set $E \subseteq \mathbb{R}^n$, $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$, $\widetilde{B}(x,r) = B(x,r) \cap \Omega$, by c,C, c_1, c_2 etc, we denote various absolute positive constants, which may have different values even in the same line.

We refer to the book [7] for variable exponent Lebesgue spaces but give some basic definitions and facts. Let $p(\cdot)$ be a measurable function on Ω with values in $(1,\infty)$. An open set Ω which may be unbounded throughout the whole paper. We mainly suppose that

$$1 < p_- \le p(x) \le p_+ < \infty,$$

where $p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x)$, $p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x)$. By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int\limits_{\Omega} |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 : \; I_{p(\cdot)}\left(\frac{f}{\eta}\right) \le 1\right\},$$

this is a Banach function space. By $p'(\cdot) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$, we denote the conjugate exponent.

The space $L^{p(\cdot)}(\Omega)$ coincides with the space

$$\left\{ f(x) : \left| \int\limits_{\Omega} f(y)g(y)dy \right| < \infty \quad for\ all \quad g \in L^{p'(\cdot)}(\Omega) \right\}$$

up to the equivalence of the norms

$$\|f\|_{L^{p(\cdot)}(\Omega)} \approx \sup_{\|g\|_{L^{p'(\cdot)}} \le 1} \left| \int\limits_{\Omega} f(y)g(y)dy \right|$$

see [19], or [20].

For the basics on variable exponent Lebesgue spaces we refer to [19].

$\mathcal{P}(\Omega)$ is the set of bounded measurable functions $p : \Omega \to [1, \infty)$;
$\mathcal{P}^{log}(\Omega)$ is the set of exponents $p \in \mathcal{P}(\Omega)$ satisfying the local log-condition

$$|p(x) - p(y)| \leq \frac{A}{-\ln|x-y|}, \quad |x-y| \leq \frac{1}{2} \ \ x, y \in \Omega,$$

where $A = A(p) > 0$ does not depend on x, y;
$\mathbb{P}^{log}(\Omega)$ is the set of exponents $p \in \mathcal{P}^{log}(\Omega)$ with $1 < p_- \leq p_+ < \infty$;
for Ω which may be unbounded, by $\mathcal{P}_\infty(\Omega)$, $\mathcal{P}^{log}_\infty(\Omega)$, $\mathbb{P}^{log}_\infty(\Omega)$ we denote the subsets of the above sets of exponents satisfying the decay condition (when Ω is unbounded)

$$|p(x) - p(\infty)| \leq \frac{A_\infty}{\ln(2 + |x|)}, \quad x \in \mathbb{R}^n.$$

where $p_\infty = \lim\limits_{x \to \infty} p(x) > 1$.

We will also make use of the estimate provided by the following lemma (see [7], Corollary 4.5.9).

$$\|\chi_{\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \leq C r^{\theta_p(x,r)}, \qquad x \in \Omega, \ p \in \mathbb{P}^{log}_\infty(\Omega),$$

where $\theta_p(x,r) = \begin{cases} \frac{n}{p(x)}, & r \leq 1, \\ \frac{n}{p(\infty)}, & r \geq 1 \end{cases}$.

Let $\lambda(x)$ be a measurable function on Ω with values in $[0, n]$. The variable Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ is defined as the set of integrable functions f on Ω with the finite norms

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x \in \Omega, \ t > 0} t^{-\frac{\lambda(x)}{p(x)}} \|f \chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)},$$

respectively.

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r > 0} |B(x,r)|^{-1} \int\limits_{\widetilde{B}(x,r)} |f(y) - f_{\widetilde{B}(x,r)}| dy,$$

where $f_{\widetilde{B}(x,t)}(x) = |\widetilde{B}(x,t)|^{-1} \int\limits_{\widetilde{B}(x,t)} f(y) dy$.

Definition 1. *We define the $BMO(\Omega)$ space as the set of all locally integrable functions f with finite norm*

$$\|f\|_{BMO} = \sup_{x \in \Omega} M^\sharp f(x) = \sup_{x \in \Omega, \ r > 0} |B(x,r)|^{-1} \int\limits_{\widetilde{B}(x,r)} |f(y) - f_{\widetilde{B}(x,r)}| dy.$$

Definition 2. *We define the $BMO_{p(\cdot)}(\Omega)$ space as the set of all locally integrable functions f with finite norm*

$$\|f\|_{BMO_{p(\cdot)}} = \sup_{x\in\Omega,\, r>0} \frac{\|(f(\cdot) - f_{\widetilde{B}(x,r)})\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)}}{\|\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)}}.$$

Theorem 1. [21] *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_{\infty}(\Omega)$. Then the norms $\|\cdot\|_{BMO_{p(\cdot)}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

By $VMO(\mathbb{R}^n)$, we denote the BMO-closure of the space $C_0^\infty(\mathbb{R}^n)$, where $C_0^\infty(\mathbb{R}^n)$ is the set of all functions from $C_0^\infty(\mathbb{R}^n)$ with compact support.

Everywhere in the sequel the functions $\varphi(x,r)$, $\varphi_1(x,r)$ and $\varphi_2(x,r)$ used in the body of the paper, are positive measurable functions on $\Omega \times (0,\infty)$. We find it convenient to define the generalized weighted Morrey spaces in the form as follows.

Definition 3. *Let $1 \le p(x) < \infty$, $x \in \Omega$. The variable exponent generalized Morrey space $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$ is defined as the set of integrable functions f on Ω with the finite norms*

$$\|f\|_{\mathcal{M}^{p(\cdot),\varphi}} = \sup_{x\in\Omega, r>0} \frac{1}{\varphi(x,r) r^{\theta_p(x,r)}} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))},$$

respectively. According to this definition, we recover the space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the choice $\varphi(x,r) = r^{\theta_p(x,r) - \frac{\lambda(x)}{p(x)}}$:

$$\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega) = \mathcal{M}^{p(\cdot),\varphi(\cdot)}(\Omega)\Big|_{\varphi(x,r) = r^{\theta_p(x,r) - \frac{\lambda(x)}{p(x)}}}.$$

Everywhere in the sequel we assume that

$$\sup_{x\in\Omega,\, r>0} \frac{r^{\theta_p(x,r)}}{\varphi(x,r)} < \infty$$

which makes the space $\mathcal{M}^{p(\cdot),\varphi(\cdot)}(\Omega)$ nontrivial.

The following theorem is valid.

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_{\infty}(\Omega)$, $b \in BMO(\Omega)$, the functions $\varphi_1(x,r)$ and $\varphi_2(x,r)$ satisfy the condition*

$$\int_t^\infty \left(1 + \ln\frac{s}{t}\right) \varphi_1(x,s) \frac{ds}{s} \le C\varphi_2(x,t). \tag{1}$$

Then the operator $[b,T]$ is bounded from the space $\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$ to the space $\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)$.

In the proof of Theorem , we need the following characterization that a subset in $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$ is a strongly pre-compact set, which is in itself interesting.

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_{\infty}(\Omega)$. Suppose $\mathcal{W}$ is a subset in $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$ satisfying the following conditions:*

i) Norm boundedness uniformly is

$$\sup_{f\in\mathcal{W}} \|f\|_{\mathcal{M}^{p(\cdot),\varphi}(\Omega)} < \infty.$$

ii) Translation continuity uniformly is

$$\lim_{y\longrightarrow 0} \|f(\cdot+y) - f(\cdot)\|_{\mathcal{M}^{p(\cdot),\varphi}(\Omega)} = 0 \ \ \textit{for any } f \in \mathcal{W}.$$

iii) Uniformly convergence at infinity is

$$\lim_{\gamma\longrightarrow\infty} \left\|f\chi_{B(0,\gamma)}\right\|_{\mathcal{M}^{p(\cdot),\varphi}(\Omega)} = 0 \quad \textit{for any } f \in \mathcal{W},$$

where $\mathcal{W}$ is a precompact set in $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$.

Now we obtain sufficient conditions for the commutator $[b,T]$ to be a compact operator on $\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$.

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_{\infty}(\Omega)$, $b \in VMO(\Omega)$, the functions φ_1 and φ_2 satisfy the condition (1).*

Then the operator $[b,T]$ is a compact from the space $\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$ to the space $\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)$.

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